Sufficient Lie Algebraic Conditions for Sampled-Data Feedback Stabilization

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Abstract— **For nonlinear affine in the control systems, a Lie algebraic sufficient condition for sampled-data feedback semi-global stabilization is established. We use this result, in order to derive sufficient conditions for sampled-data feedback stabilization for a couple of three-dimensional systems.**

I. Introduction

Significant results towards stabilization of nonlinear systems by means of sampled-data feedback control have appeared in the literature (see for instance [1], [2], [4]-[8], [10]-[15], [17], [20] and relative references therein). In the recent works [22] and [23], the concept of *Weak Global Asymptotic Stabilization by Sampled-Data Feedback* (SDF-WGAS) is introduced for autonomous systems:

$$\begin{gathered}\dot{x} = f(x,u),\ (x,u) \in \mathbb{R}^n \times \mathbb{R}^m, \\ f(0,0) = 0\end{gathered} \tag{1.1}$$

and Lyapunov-like sufficient characterizations of this property are examined. Particularly, in [23, Proposition 2], a Lie algebraic sufficient condition for SDF-WGAS is established for the case of affine in the control single-input systems

$$\begin{gathered}\dot{x} = f(x) + ug(x),\ \ (x,u) \in \mathbb{R}^n \times \mathbb{R}, \\ f(0) = 0\end{gathered} \tag{1.2}$$

This condition constitutes an extension of the well-known "Artstein-Sontag" sufficient condition for asymptotic stabilization of systems (1.2) by means of an almost smooth feedback; (see [3], [19] and [21]).

Throughout the paper we adopt the following notations. For any pair of C^1 mappings $X : \mathbb{R}^n \to \mathbb{R}^k$, $Y : \mathbb{R}^k \to \mathbb{R}^\ell$ we denote $XY := (DY)X$, DY being the derivative of Y. By $[\cdot,\cdot]$ we denote the Lie bracket operator, namely, $[X,Y] = XY - YX$ for any pair of C^1 mappings $X, Y : \mathbb{R}^n \to \mathbb{R}^n$.

The precise statement of [23, Proposition 2] is the following. Assume that $f, g \in C^2$ and there exists a C^2, positive definite and proper function $V : \mathbb{R}^n \to \mathbb{R}^+$ such that the following implication holds:

$$\begin{aligned}&(gV)(x) = 0, x \neq 0 \\ &\Rightarrow \begin{cases} either\ (fV)(x) < 0, \\ \qquad (\text{“Artstein} - \text{Sontag” condition}) \\ or\ (fV)(x) = 0;\ ([f,g]V)(x) \neq 0\end{cases}\end{aligned} \tag{1.3}$$

Then system (1.2) is SDF-WGAS.

In the present work, we first deal with the general case (1.1), providing a Lyapunov characterization for a stronger version of SDF-WGAS. Particularly, Proposition 2 of our work asserts that for systems (1.1) the same Lyapunov characterization of SDF-WGAS, originally proposed in [22] (see Assumption 1 below), implies *Semi-Global Asymptotic Stabilization by means of a time-varying Sampled-Data Feedback* (SDF-SGAS), which is a stronger type of SDF-WGAS. We exploit the result of Proposition 2 to establish in Proposition 3 a Lie algebraic sufficient condition for SDF-SGAS(WGAS) for systems (1.2), much weaker than (1.3).

The result of Proposition 3 is then used, in order to study the SDF-SGAS for a couple of 3-dimensional affine in the control cases (Corollaries 1 and 2).

The precise statements of Propositions 2 and 3 and of Corollaries 1 and 2 are given in Section II. Proofs of both corollaries are given in Section III. Detailed proofs of Propositions 2 and 3 can be found in [24]. For completeness, an outline of proof of Proposition 3 is also provided in Section III.

II. Definitions and Main Results

Consider system (1.1) and assume that $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ is Lipschitz continuous. We denote by $x(\cdot) = x(\cdot, s, x_0, u)$ the trajectory of (1.1) with initial condition $x(s, s, x_0, u) = x_0 \in \mathbb{R}^n$ corresponding to certain measurable and locally essentially bounded control $u : [s, T_{\max}) \to \mathbb{R}^m$, where $T_{\max} = T_{\max}(s, x_0, u)$ is the corresponding maximal existing time of the trajectory.

Definition 1: We say that system (1.1) is *Weakly Globally Asymptotically Stabilizable by Sampled-Data Feedback* (SDF-WGAS), if for any constant $\tau > 0$ there exist mappings $T : \mathbb{R}^n \backslash \{0\} \to \mathbb{R}^+ \backslash \{0\}$ satisfying

$$T(x) \leq \tau,\ \ \forall x \in \mathbb{R}^n \setminus \{0\} \tag{2.1}$$

and $k(t, x; x_0) : \mathbb{R}^+ \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^m$ such that for any fixed $(x, x_0) \in \mathbb{R}^n \times \mathbb{R}^n$ the map $k(\cdot, x; x_0) : \mathbb{R}^+ \to \mathbb{R}^m$ is measurable and locally essentially bounded and such that for every $x_0 \neq 0$ there exists a sequence of times

$$t_1 := 0 < t_2 < t_3 < \ldots < t_\nu < \ldots\ \ \text{, with } t_\nu \to \infty \tag{2.2}$$

in such a way that the trajectory $x(\cdot)$ of the sampled-data closed loop system:

$$\begin{aligned}\dot{x} = f(x, k(t, x(t_i); x_0)),\ t \in [t_i,\ t_{i+1}),\ \ i = 1, 2, \ldots \\ x(0) = x_0 \in \mathbb{R}^n\end{aligned} \tag{2.3}$$

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satisfies:

$$t_{i+1} - t_i = T(x(t_i)),\ i = 1, 2, \dots \tag{2.4}$$

and the following properties:

$$\text{Stability:}\quad \begin{array}{c}\forall \varepsilon > 0 \Rightarrow \exists \delta = \delta(\varepsilon) > 0 : |x(0)| \le \delta \\ \Rightarrow |x(t)| \le \varepsilon,\ \forall t \ge 0\end{array} \tag{2.5}$$

$$\text{Attractivity:}\quad \lim_{t\to\infty} x(t) = 0,\ \forall x(0) \in \mathbb{R}^n \tag{2.6}$$

where $|x|$ denotes the Euclidean norm of the vector x.

Next we give the Lyapunov characterization of SDF-WGAS proposed in [22] and [23], that constitutes a generalization of the concept of the *control Lyapunov function* (see Definition 5.7.1 in [18]).

Assumption 1: *There exist a positive definite C^0 function $V : \mathbb{R}^n \to \mathbb{R}^+$ and a function $a \in K$ (namely, $a(\cdot)$ is continuous, strictly increasing with $a(0) = 0$) such that for every $\xi > 0$ and $x_0 \neq 0$ there exists a constant $\varepsilon = \varepsilon(x_0) \in (0, \xi]$ and a measurable and locally essentially bounded control $u(\cdot, x_0) : [0, \varepsilon] \to \mathbb{R}^m$ satisfying*

$$V(x(\varepsilon, 0, x_0, u(\cdot, x_0))) < V(x_0); \tag{2.7a}$$

$$V(x(s, 0, x_0, u(\cdot, x_0))) \le a(V(x_0)),\ \ \forall s \in [0, \varepsilon] \tag{2.7b}$$

The following result was established in [22].

Proposition 1: Under Assumption 1, system (1.1) is SDF-WGAS.

We now present the concept of SDF-SGAS, which, as mentioned above, is a strong version of SDF-WGAS:

Definition 2: We say that system (1.1) is *Semi-Globally Asymptotically Stabilizable by Sampled-Data Feedback* (SDF-SGAS), if for every $R > 0$ and for any given partition of times

$$T_1 := 0 < T_2 < T_3 < \dots < T_\nu < \dots\ \text{ with } T_\nu \to \infty \tag{2.8}$$

there exist a neighborhood Π of zero with $B[0, R] := \{x \in \mathbb{R}^n : |x| \le R\} \subset \Pi$ and a map $k : \mathbb{R}^+ \times \Pi \to \mathbb{R}^m$ such that for any $x \in \Pi$ the map $k(\cdot, x) : \mathbb{R}^+ \to \mathbb{R}^m$ is measurable and locally essentially bounded and the trajectory $x(\cdot)$ of the sampled-data closed loop system

$$\begin{array}{c}\dot{x} = f(x, k(t, x(T_i))),\ t \in [T_i,\ T_{i+1}),\ \ i = 1, 2, \dots \\ x(0) \in \Pi\end{array} \tag{2.9}$$

satisfies:

$$\text{Stability:}\quad \begin{array}{c}\forall \varepsilon > 0 \Rightarrow \exists \delta = \delta(\varepsilon) > 0 : x(0) \in \Pi, \\ |x(0)| \le \delta \Rightarrow |x(t)| \le \varepsilon,\ \forall t \ge 0\end{array} \tag{2.10}$$

$$\text{Attractivity:}\quad \lim_{t\to\infty} x(t) = 0,\ \forall x(0) \in \Pi \tag{2.11}$$

It should be pointed out that Definition 2 is stronger than the concept of sampled-data semi-global asymptotic stabilization adopted in earlier relative works in the literature, because the partition of times in (2.8) is arbitrary.

The proof of the following proposition is based on a generalization of the methodology applied in [22] and is provided in [24]:

Proposition 2: Under Assumption 1, system (1.1) is SDF-SGAS and therefore SDF-WGAS.

We next present the statement of the central result of present work, which provides a Lie algebraic sufficient condition for SDF-SGAS(WGAS) for the affine in the control single-input system (1.2). In the sequel we assume that its dynamics f, g are smooth (C^∞). We denote by $Lie\{f, g\}$ the Lie algebra generated by $\{f, g\}$. Also, let $L_1 := span\{f, g\}$ and $L_{i+1} := span\{[X, Y],\ X \in L_i, Y \in L_1\},\ i = 1, 2, \dots$ and for any nonzero $\Delta \in Lie\{f, g\}$ define

$$order_{\{f,g\}}\Delta \begin{cases} := 1, \text{ if } \Delta \in L_1 \setminus \{0\} \\ := k > 1, \text{if } \Delta = \Delta_1 + \Delta_2, \\ \quad \text{with } \Delta_1 \in L_k \setminus \{0\} \text{ and} \\ \quad \Delta_2 \in span\{\cup_{i=1}^{i=k-1} L_i\} \end{cases} \tag{2.12}$$

As a consequence of Proposition 2 we get:

Proposition 3: For the affine in the control case (1.2) assume that there exists a smooth function $V : \mathbb{R}^n \to \mathbb{R}^+$, being positive definite and proper, such that for every $x \neq 0$, either $(gV)(x) \neq 0$, or one of the following properties hold: Either

$$(gV)(x) = 0 \Rightarrow (fV)(x) < 0 \tag{2.13}$$

or there exists an integer $N = N(x) \ge 1$ such that

$$(gV)(x) = 0,\ (f^iV)(x) = 0,\ \ i = 1, 2, \dots, N \tag{2.14a}$$

$$\begin{array}{c}(\Delta_1\Delta_2 \dots \Delta_k V)(x) = 0 \\ \forall \Delta_1, \Delta_2, \dots, \Delta_k \in Lie\{f, g\} \setminus \{g\} \\ \text{with } \displaystyle\sum_{p=1}^{k} order_{\{f,g\}}\Delta_p \le N\end{array} \tag{2.14b}$$

where $(f^iV)(x) := f(f^{i-1}V)(x),\ i = 2, 3, \dots$, $(f^1V)(x) := (fV)(x)$ and in such a way that one of the following properties hold:

(P1) $$(f^{N+1}V)(x) < 0 \tag{2.15}$$

(P2) N is odd and

$$([\underbrace{[\dots[[f, g], g], \dots, g], g]}_{N} V)(x) \neq 0 \tag{2.16}$$

(P3) N is even and

$$([\underbrace{[\dots[[f, g], g], \dots, g], g]}_{N} V)(x) < 0 \tag{2.17}$$

(P4) N is an arbitrary positive integer with

$$(f^{N+1}V)(x) = 0, \tag{2.18a}$$

$$([\underbrace{[\dots[[g, f], f], \dots, f], f]}_{N} V)(x) \neq 0 \tag{2.18b}$$

Then system (1.2) is SDF-SGAS and therefore SDF-WGAS.

Remark 1: For the particular case of $N = 1$, condition (2.14a) is equivalent to $(gV)(x) = 0$ and $(fV)(x) = 0$, the previous equality is equivalent to (2.14b) and obviously (2.16) is equivalent to $([f, g]V)(x) \neq 0$. It turns out, according to the statement of Proposition 3, that, under (1.3), the system (1.2) is SDF-SGAS and therefore SDF-WGAS; the latter conclusion, namely, that (1.3) implies SDF-WGAS, is the precise statement of [23, Proposition 2].

An interesting consequence of Proposition 3 concerning 3-dimensional systems (1.2) is the following result:

Corollary 1: Consider the 3-dimensional system (1.2) and assume that:

(I) $$span\{g(x), [f, g](x), [f, [f, g]](x)\} = \mathbb{R}^3 \tag{2.19}$$

(II) There exists a smooth positive definite and proper function $V : \mathbb{R}^n \to \mathbb{R}^+$ such that

$$DV(x) \neq 0,\ \forall x \neq 0 \tag{2.20}$$

and in such a way that, either (2.13) holds, or

$$(gV)(x) = 0 \Rightarrow (f^i V)(x) = 0,\ \forall x \neq 0,\ i = 1, 2, 3 \tag{2.21}$$

Then the system is SDF-SGAS.

We finally consider the following interesting case of 3-dimensional systems:

$$\dot{x}_1 = a(x_1, x_2, x_3)x_3^L,\ \dot{x}_2 = b(x_1, x_2, x_3)x_3,\ \dot{x}_3 = u,\\ (x_1, x_2, x_3) \in \mathbb{R}^3 \tag{2.22}$$

where

$$L \geq 3 \text{ is a positive odd integer} \tag{2.23}$$

and the functions $a, b : \mathbb{R}^3 \to \mathbb{R}$ are smooth (C^∞) and satisfy

$$a(x), b(x) \neq 0,\ \forall x \in \mathbb{R}^3 \tag{2.24}$$

It can be easily verified that (2.22) does not satisfy the well known Brockett's condition for smoothly static feedback stabilization. For $a(\cdot) = b(\cdot) = 1$, it was established in [9], that (2.22) is small time locally controllable and in [16] that is *locally* asymptotically stabilizable by means of a smooth time-varying periodic feedback. We use the result of Proposition 3 of present work to establish the following result.

Corollary 2: Under hypotheses (2.23) and (2.24), system (2.22) is SDF-SGAS.

III. Proofs

Outline of proof of Proposition 3: (As mentioned, the complete proof is found in [24]) Let $0 \neq x_0 \in \mathbb{R}^n$ and suppose first that, either $(gV)(x_0) \neq 0$, or (2.13) is fulfilled, namely, $(gV)(x_0) = 0$ and $(fV)(x_0) < 0$. Then there exists a constant input u such that both (2.7a) and (2.7b) hold; particularly, for every sufficiently small $\varepsilon > 0$ we have:

$$V(x(s, 0, x_0, u)) < V(x_0), \forall s \in (0, \varepsilon] \tag{3.1}$$

Assume next that there exists an integer $N = N(x_0) \geq 1$ satisfying (2.14), as well as one of the properties (P1), (P2), (P3), (P4) with $x = x_0$. Then (2.14) implies:

$$(fV)(x_0) = (gV)(x_0) = 0 \tag{3.2}$$

In order to derive the desired conclusion, we proceed as follows. Define:

$$X := f + u_1 g,\ Y := f + u_2 g \tag{3.3}$$

and let us denote by $X_t(z)$ and $Y_t(z)$ the trajectories of the systems $\dot{x} = X(x)$ and $\dot{y} = Y(y)$, respectively, initiated at time $t = 0$ from some $z \in \mathbb{R}^n$. Also, for any constant $\rho > 0$ define:

$$R(t) := (X_{\rho t} \circ Y_t)(x_0),\ t \geq 0,\ R(0) = x_0 \tag{3.4a}$$
$$m(t) := V(R(t)), t \geq 0 \tag{3.4b}$$

and denote in the sequel by $\overset{(\nu)}{m}(\cdot)$, $\nu = 1, 2, ...$ its ν-time derivative. By taking into account (3.2)-(3.4) and exploiting the Campbell-Baker-Hausdorff formula for the right hand side map of (3.4a), together with an induction procedure, it can be shown that

$$\overset{(1)}{m}(0) = 0 \tag{3.5a}$$

and for every integer $n \geq 2$, the n-time derivative $\overset{(n)}{m}(\cdot)$ of $m(\cdot)$ satisfies

$$\begin{aligned} &\overset{(n)}{m}(0) \in (A_0^n V)(x_0) \\ &+ span \left\{ \begin{array}{l} \rho^{r_n}(A_{i_1} A_{i_2} ... A_{i_\nu} V)(x_0) : \nu \geq 2; \\ i_1, i_2, ... i_\nu \in \mathbb{N}_0; \sum_{j=1}^{\nu} order_{\{X,Y\}} A_{i_j} = n; \\ r_n = \sum_{j=1}^{\nu} i_j \in \{1, 2, ..., n-2\} \end{array} \right\} \\ &+ \rho^{n-1}(A_{n-1} V)(x_0) \end{aligned} \tag{3.5b}$$

where

$$\begin{aligned} &A_0 := \rho X + Y, \\ &A_\nu := [...[[Y, \underbrace{X], X], \ldots, X]}_{\nu}, \nu = 1, 2, ... \end{aligned} \tag{3.6}$$

Since $A_\nu \in Lie\{X, Y\}$, we may define, according to (2.12), the order of each A_ν with respect to the Lie algebra of $\{X, Y\}$; particularly, in our case, we have:

$$order_{\{X,Y\}} A_\nu = \nu + 1,\ \forall \nu = 0, 1, 2, \ldots \tag{3.7}$$

By taking into account definition (3.3) of the vector fields X and Y and by setting

$$u_2 = -\rho u_1,\ \rho > 0 \tag{3.8}$$

we get

$$\begin{aligned}
&A_0 = (\rho+1)f,\ A_1 = (\rho+1)u_1[f,g],\\
&A_2 = (\rho+1)(u_1^2[[f,g],g] - u_1[[g,f],f])\\
&\quad\vdots\\
&A_n = (\rho+1)u_1^n[...[[f,\underbrace{g],g],...,g]}_{n}\\
&+(\rho+1)u_1^{n-1}([[[...[f,\underbrace{g],...,g],g}_{n-1}],f]\\
&+[[[...[f,\underbrace{g],...,g}_{n-2}],f],g]+...\\
&+[...[[[f,g],f],\underbrace{g]...,g]}_{n-2})+[...[[[f,g],f],\underbrace{g]...,g]}_{n-2})\\
&+...+(\rho+1)u_1^2([[[...[[f,g],\underbrace{f],...,f],f}_{n-2}],g]\\
&+[[[...[[f,g],\underbrace{f],...,f}_{n-3}],g],f]\\
&+...+[[...[[[f,g],g],\underbrace{f]...,f],f]}_{n-2})\\
&-(\rho+1)u_1[...[[g,\underbrace{f],f],...,f]}_{n},\ n=3,4,...
\end{aligned} \tag{3.9}$$

Obviously, (3.9) implies:

$$\begin{aligned}
A_k \in span\{\Delta \in Lie\,\{f,g\}\setminus\{g\} : order_{\{f,g\}}\Delta = k+1\}\\
k=0,1,2,\ldots
\end{aligned} \tag{3.10}$$

Also, we recall from (3.5b) and (3.7) that $r_n = \sum_{s=1}^{\nu} i_s \in \{1,2,\ldots,n-2\}$ and $\sum_{j=1}^{\nu} order_{\{X,Y\}}A_{i_j} = r_n + \nu = n$ with $\nu \geq 2$ and therefore $\nu \leq n-1$. By (3.5b)-(3.10) and the previous facts we get:

$$\begin{aligned}
&\overset{(n)}{m}(0) \in (\rho+1)^n(f^nV)(x_0) + u_1\pi_1(\rho,\rho+1;x_0)\\
&\quad + span\left\{u_1^k\pi_k(\rho,\rho+1;x_0), k=2,...,n-2\right\}\\
&\quad + \rho^{n-1}(\rho+1)u_1^{n-1}([\ldots[[f,\underbrace{g],g],\ldots,g]}_{n-1}V)(x_0)\\
&\quad - \rho^{n-1}(\rho+1)u_1([\ldots[[g,\underbrace{f],f],\ldots,f]}_{n-1}V)(x_0)
\end{aligned} \tag{3.11}$$

for $n = 2,3,...$ and for certain smooth functions $\pi_k : \mathbb{R}^2 \times \mathbb{R}^n \to \mathbb{R},\ k=1,2,\ldots,n-2$ satisfying the following properties:

(S1) For every $x_0 \in \mathbb{R}^n$, each map $\pi_k(\alpha,\beta;x_0) : \mathbb{R}^2 \to \mathbb{R}$ is a polynomial with respect to the first two variables in such a way that

$$\begin{aligned}
&span\{\pi_k(\alpha,\beta;x_0),\ k=1,2,\ldots,n-2\} \subset\\
&span\{(\Delta_1\Delta_2...\Delta_i V)(x_0);\ i\in\mathbb{N},\\
&\quad \Delta_1,\Delta_2,...,\Delta_i \in Lie\{f,g\}\backslash\{g\};\\
&\quad \textstyle\sum_{j=1}^{j=i} order_{\{f,g\}}\Delta_j = n\ \}
\end{aligned} \tag{3.12}$$

(S2) For each $x_0 \in \mathbb{R}^n$ there exist integers $\lambda_i,\ \mu_i,\ i = 1,2,...,\ L \in \mathbb{N}$ with $1 \leq \lambda_i \leq n-2,\ 2 \leq \mu_i \leq n-1$ such that the map $\pi_1(\alpha,\beta;x_0) : \mathbb{R}^2 \to \mathbb{R}$ satisfies:

$$\pi_1(\alpha,\beta;x_0) \in span\left\{\alpha^{\lambda_1}\beta^{\mu_1}, \alpha^{\lambda_2}\beta^{\mu_2}, ..., \alpha^{\lambda_L}\beta^{\mu_L}\right\}$$

The latter implies that for each fixed $x_0 \in \mathbb{R}^n$ the polynomials $\pi_1(\rho,\rho+1;x_0)$ and

$$-\rho^{n-1}(\rho+1)([\ldots[[g,\underbrace{f],f],\ldots,f]}_{n-1}V)(x_0)$$

are linearly independent, provided that

$$([[\ldots[[g,\underbrace{f],f],\ldots,f],f]}_{n-1}V)(x_0) \neq 0 \tag{3.13}$$

If we define:

$$\begin{aligned}
\xi_n(\rho;x) :=&\pi_1(\rho,\rho+1;x_0)\\
&-\rho^{n-1}(\rho+1)([\ldots[[g,\underbrace{f],f],\ldots,f]}_{n-1}V)(x_0)
\end{aligned} \tag{3.14}$$

the inclusion (3.11) is rewritten:

$$\begin{aligned}
&\overset{(n)}{m}(0) \in (\rho+1)^n(f^nV)(x_0) + u_1\xi_n(\rho;x_0)\\
&+ span\left\{u_1^k\pi_k(\rho,\rho+1;x_0), k=2,...,n-2\right\}\\
&+ \rho^{n-1}(\rho+1)u_1^{n-1}([\ldots[[f,\underbrace{g],g],\ldots,g]}_{n-1}V)(x_0)
\end{aligned} \tag{3.15}$$

and a constant $\rho = \rho(x_0) > 0$ can be found with

$$\xi_n(\rho;x_0) \neq 0 \tag{3.16}$$

provided that (3.13) holds. Suppose now that there exists an integer $N = N(x_0) \geq 1$ satisfying (2.14), as well as one of the properties (P1), (P2), (P3), (P4) with $x = x_0$. By (3.5a) and by taking into account (2.14), (3.11) and (3.12) it follows:

$$\overset{(n)}{m}(0) = 0,\ n=1,2,\ldots,N \tag{3.17}$$

and also, by taking into account (3.4a), (3.4b) and (3.8)-(3.16) it can be shown that in all cases considered in the statement of Proposition 3, there exist constants $\rho = \rho(x_0) > 0$ and u_1 such that, if we define

$$w(s;t,x_0) := \left\{\begin{array}{c} u_2 = -\rho u_1,\ s\in[0,t]\\ u_1,\ s\in(t,t+\rho t]\end{array}\right. \tag{3.18}$$

it holds $\overset{(N+1)}{m}(0) < 0$, which, in conjunction with (3.17), asserts that for every sufficiently small $\sigma = \sigma(x_0) > 0$ we have

$$m(t) < m(0),\ \forall t \in (0,\sigma] \tag{3.19a}$$

where

$$\begin{aligned}
m(t) :&= V((X_{\rho t}\circ Y_t)(x_0))\\
&= V(x(t+\rho t, 0, x_0, w(\cdot;t,x_0))
\end{aligned} \tag{3.19b}$$

and $x(\cdot,0,x_0,w(\cdot;t,x_0))$ is the trajectory of (1.2) corresponding to the input $w(\cdot;t,x_0)$. Equivalently:

$$V(x(t,0,x_0,w(\cdot;\tfrac{t}{1+\rho},x_0))) < V(x_0), \forall t \in (0,\tfrac{\sigma}{1+\rho}] \tag{3.20}$$

hence, we may pick $\varepsilon \in (0,\frac{\sigma}{1+\rho}]$ sufficiently small in such a way that inequality in (3.20) holds for $t := \varepsilon$, namely,

$$V(x(\varepsilon,0,x_0,u(\cdot,x_0))) < V(x_0) \tag{3.21a}$$

with $u(s,x_0) := w(s;\frac{\varepsilon}{1+\rho},x_0),\ s \in (0,\varepsilon]$ and simultaneously

$$V(x(s,0,x_0,u(\cdot,x_0))) \le 2V(x_0),\ \forall s \in (0,\varepsilon] \quad (3.21b)$$

We conclude, by taking into account (3.1) and (3.21), that for every $x_0 \neq 0$ and $\xi > 0$, there exist $\varepsilon = \varepsilon(x_0) \in (0,\xi]$ and a measurable and locally essentially bounded control $u(\cdot,x_0) : [0,\varepsilon] \to \mathbb{R}$ such that (2.7a) and (2.7b) hold with $a(s) := 2s$. Therefore, according to Proposition 2, (1.2) is SDF-SGAS. ■

Proof of Corollary 1: First, by invoking assumptions (2.19) and (2.20) it follows that for every $x \neq 0$, either $(gV)(x) \neq 0$, or

$$(gV)(x) = 0 \quad (3.22)$$

which in conjunction with (2.13) implies the desired statement. Also, by virtue of (2.19)-(2.21), we have

$$(fV)(x) = (f^2V)(x) = (f^3V)(x) = 0 \quad (3.23a)$$

$$|([f,g]V)(x)| + |([f,[f,g]]V)(x)| \neq 0 \quad (3.23b)$$

For those $x \neq 0$ for which (3.22) holds, we consider two cases. The first is $([f,g]V)(x) \neq 0$, which in conjunction with (3.22) and (3.23a) assert that (2.14a) and (P4) hold with $N = 1$. The other case is

$$([f,g]V)(x) = 0 \quad (3.24a)$$

$$([f,[f,g]]V)(x) \neq 0 \quad (3.24b)$$

which in conjunction with (3.22) and (3.23a) assert that (2.14a), (2.14b) and (P4) are fulfilled with $N = 2$. We conclude, according to the statement of Proposition 3, that the 3-dimensional system (1.2) is SDF-SGAS.■

Proof of Corollary 2: We define:

$$\begin{aligned} f(x) &:= (a(x_1,x_2,x_3)x_3^L,\ b(x_1,x_2,x_3)x_3,0)^T, \\ g(x) &:= (0,0,1)^T,\ x := (x_1,x_2,x_3)^T \end{aligned} \quad (3.25)$$

and

$$V(x) := \tfrac{1}{2}x_1^2 + \tfrac{1}{L+1}x_2^{L+1} + \tfrac{1}{2}x_3^2 \quad (3.26)$$

that obviously is positive definite and proper. According to the previous definitions, it follows that

$$([f,g])(x) = \begin{pmatrix} -\frac{\partial a}{\partial x_3}(x_1,x_2,x_3)x_3^L - La(x_1,x_2,x_3)x_3^{L-1} \\ -\frac{\partial b}{\partial x_3}(x_1,x_2,x_3)x_3 - b(x_1,x_2,x_3) \\ 0 \end{pmatrix} \quad (3.27a)$$

and for each integer $k : 2 \le k \le L$ it holds:

$$\begin{aligned} ([...[[f,\underbrace{g],g]...g}_{k}])(x) = (&A_{1,k}(x_1,x_2,x_3) \\ &+ (-1)^k \prod_{i=0}^{k-1}(L-i)a(x_1,x_2,x_3)x_3^{L-k}, \\ &A_{2,k}(x_1,x_2,x_3) + (-1)^k k\frac{\partial^{k-1}b}{\partial x_3^{k-1}}(x_1,x_2,x_3),0)^T \end{aligned} \quad (3.27b)$$

$$([...[[g,\underbrace{f],f]...f}_{k}])(x) = (B_{1,k}(x_1,x_2,x_3),B_{2,k}(x_1,x_2,x_3),0)^T \quad (3.27c)$$

for certain smooth functions $A_{1,k}, A_{2,k}, B_{1,k}, B_{2,k} : \mathbb{R}^3 \to \mathbb{R}$, satisfying

$$A_{1,k}(\cdot,\cdot,0) = A_{2,k}(\cdot,\cdot,0) = B_{1,k}(\cdot,\cdot,0) = B_{2,k}(\cdot,\cdot,0) = 0, \quad (3.28a)$$

and

$$\begin{aligned} &\frac{\partial^j A_{1,n}}{\partial x_2^j}(\cdot,\cdot,0) = \frac{\partial^j B_{1,n}}{\partial x_2^j}(\cdot,\cdot,0) = \frac{\partial^j B_{2,n}}{\partial x_2^j}(\cdot,\cdot,0) = 0, \\ &j = 1,...,L-1;\ n = 2,...,L-j+1; \end{aligned} \quad (3.28b)$$

From (3.25)-(3.27) we also get

$$(gV)(x) = x_3; \quad (3.29a)$$

$$\begin{aligned} ([f,g]V)(x) = &-\tfrac{\partial a}{\partial x_3}(x_1,x_2,x_3)x_1x_3^L \\ &- La(x_1,x_2,x_3)x_1x_3^{L-1} \\ &- \tfrac{\partial b}{\partial x_3}(x_1,x_2,x_3)x_2^Lx_3 - b(x_1,x_2,x_3)x_2^L,\ \forall x \in \mathbb{R}^3 \end{aligned} \quad (3.29b)$$

and for any integer k: $2 \le k \le L$ it holds:

$$\begin{aligned} ([...[[f,\underbrace{g],g]...g}_{k}]V)(x) = &A_{1,k}(x_1,x_2,x_3)x_1 \\ &+ (-1)^k \prod_{i=0}^{k-1}(L-i)a(x_1,x_2,x_3)x_3^{L-k}x_1 \\ &+ A_{2,k}(x_1,x_2,x_3)x_2^L + (-1)^k k\frac{\partial^{k-1}b}{\partial x_3^{k-1}}(x_1,x_2,x_3)x_2^L \end{aligned} \quad (3.29c)$$

$$\begin{aligned} ([...[[g,\underbrace{f],f]...f}_{k}]V)(x) = &B_{1,k}(x_1,x_2,x_3)x_1 \\ &+ B_{2,k}(x_1,x_2,x_3)x_2^L \end{aligned} \quad (3.29d)$$

Let $x \neq 0$ for which

$$(gV)(x) = x_3 = 0 \quad (3.30)$$

It then follows by virtue of (3.25), (3.26) and (3.29a) that

$$(f^kV)(x) = 0,\ k = 1,2,... \quad (3.31)$$

therefore (2.14a) holds, and further, by invoking (2.23)

$$([f,g]V)(x) = -b(x_1,x_2,0)x_2^L \quad (3.32)$$

Then we may distinguish the following two cases:

Case 1: $x_2 \neq 0$ with $x_1 \neq 0$ and $x_3 = 0$. Then by taking into account our hypothesis (2.24) and (3.32), it follows that $([f,g]V)(x) \neq 0$, which in conjunction with (3.31) asserts that both (2.14) and (P2) in the statement of Proposition 3 are satisfied with $N = 1$.

Case 2: $x_2 = 0$ with $x_1 \neq 0$ and $x_3 = 0$. It then follows from (2.24), (3.29c), (3.30) and (3.32) that

$$([[[f,\underbrace{g],g],...,g}_{k}]V)(x) = 0,\ k = 1,...,L-1; \quad (3.33a)$$

$$([...[[f,\underbrace{g],g],...,g}_{L}]V)(x) \neq 0, \forall x_1 \neq 0 \quad (3.33b)$$

and therefore we can easily verify from our hypotheses (2.23), (2.24) and (3.33b), that (P2) holds with $N = L$. By

taking into account (3.31), it also follows that (2.14a) holds for $k = 1, ..., L$, thus, in order to verify that all statements of Proposition 3 are satisfied, it remains to show that (2.14b) holds as well. Particularly, we show that, if we define

$$\pi_k(x) := (\Delta_1\Delta_2...\Delta_k V)(x);$$
$$\Delta_1, ..., \Delta_k \in Lie\{f, g\}\backslash\{g\} \text{ with } \sum_{p=1}^{k} order_{\{f,g\}}\Delta_p \leq L \tag{3.34}$$

it holds

$$\pi_k(x_1, 0, 0) = 0, \ \forall x_1 \in \mathbb{R} \tag{3.35}$$

In order to establish (3.35), it suffices to consider in (3.34) only those Δ_p satisfying

$$\Delta_p \in \{f, \ [...[[f, \underbrace{g], g], ..., g}_{k_1}], \ [[g, \underbrace{f], f], ..., f}_{k_2}]\}$$

for certain appropriate $k_1, k_2 \in \mathbb{N}$. Notice first that, due to (3.25), (3.27) and (3.28), each Δ_p, $p = 1, 2, ..., k$ above is written as

$$\Delta_p(x) = (C_{1,k}(x_1, x_2, x_3), C_{2,k}(x_1, x_2, x_3), 0)^T \tag{3.36a}$$

for all $x = (x_1, x_2, x_3) \in \mathbb{R}^3$ and for certain smooth functions $C_{1,k}(\cdot, \cdot, \cdot)$ and $C_{2,k}(\cdot, \cdot, \cdot)$ with

$$C_{1,k}(\cdot, \cdot, 0) = 0; \ \ \frac{\partial^j C_{1,q}}{\partial x_2^j}(\cdot, \cdot, 0) = 0,$$
$$j = 1, ..., k; \ q = 1, ..., k - j + 1,$$
$$\text{for the case}$$
$$\Delta_p \in D := \{[...[[f, \underbrace{g], g], ..., g}_{n}], n = 1, ..., L\} \tag{3.36b}$$

and

$$C_{1,k}(\cdot, \cdot, 0) = 0; \ C_{2,k}(\cdot, \cdot, 0) = 0; \ \ \frac{\partial^j C_{1,q}}{\partial x_2^j}(\cdot, \cdot, 0) = 0,$$
$$j = 1, ..., k; \ q = 1, ..., k - j + 1,$$
$$\text{for those } \Delta_p \in Lie\{f, g\}\backslash\{\{g\} \cup D\} \tag{3.36c}$$

We then may use the previous facts, together with (3.25)-(3.28) and an elementary induction procedure, in order to establish that for every integer $k \in \{1, ..., L-1\}$, for which the inequality in (3.34) holds, there exist smooth functions $\Xi_1 = \Xi_1(x_1, x_2, x_3)$ and $\Xi_2 = \Xi_2(x_1, x_2, x_3)$ in such a way that

$$\Xi_1(\cdot, \cdot, 0) = 0 \tag{3.37a}$$
$$\pi_k(x_1, x_2, x_3) = \Xi_1(x_1, x_2, x_3) + \Xi_2(x_1, x_2, x_3) x_2^{L-k+1} \tag{3.37b}$$

and the latter establishes (3.35). It follows from (2.24), (3.31), (3.33b) and (3.35) that for the Case 2, both (2.14) and (P2) hold with $N = L$.

We conclude, that in both Cases 1 and 2, hypothesis of Proposition 3 is satisfied, therefore system is SDF-SGAS.■

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